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Ten non-polynomial cubic splines for some classes of Fredholm integral equations

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ABSTRACT

In this paper, a new general form for the non-polynomial cubic spline function is introduced. This form enables us to generate about ten different formulas of the proposed spline function. Although some of these functions have been used before to solve some types of differential and integral equations, the others are considered new. These ten functions are used to approximate the solution of linear, nonlinear, and fuzzy Fredholm integral equations of second kind. Moreover, the convergence analysis of the proposed method is established which indicates that its order of convergence is four. Also, some examples are solved and compared with the previous methods which showed that the present method is more accurate.

1. Introduction

Consider the linear Fredholm integral equation (LFIE) of second kind [1–15] and the nonlinear Fredholm integral equation (NLFIE) [13–23] take the following forms, respectively:

$$y(x) = f(x) + \int_a^b \psi(x,t)y(t)dt, \tag{1.1}$$

$$y(x) = f(x) + \int_a^b \psi(x,t,y(t))dt, \qquad a \le x \le b, \tag{1.2}$$

where $f(x), \psi(x,t)$ and $\psi(x,t,y(t))$ are given continuous functions defined on $[a,b]$ and $y(x)$ is the unknown function. The fuzzy Fredholm integral equation (FFIE) [25–28] is represented by equation (1.1) when $\psi(x,t)$ is a bivariate function over the square $a \le x, t \le b$ and $f(x)$ is a given fuzzy function.

Many numerical techniques are used to investigate the solutions of LFIE, NLFIE and FFIE such as quarter–sweep Guass–Seidel method (QSGSM) [1], functional approximation method (FAM) [2]), quasi–interpolation method (QIM) [3], quarter–sweep iteration method (QSIM) [6]), triangular functions method (TFM) [8], collocation method (CM) [13], Newton–Kantorovich–quadrature method (NKQM) [17], discrete adomian decomposition method (DADM) [22], discrete homotopy analysis method (DHAM) [23], block–pulse functions (BPFs) [26] ... etc. For more details, Muthuvalu and Sulaiman [1] studied the solution of LFIE using QSGSM which gave better result than half and full sweep Guass–Seidel methods. Long and Nelakanti [2] used FAM to solve it while Müller and Varnhorn [3] solved it using QIM. Also, Bellour et al. [4] discussed its solution using natural cubic spline approximation and cubic spline quasi–interpolation method. Panda et al. [5] introduced a modified approach depends on the quadrature rule to approximate the integral in equation (1.1) and the Lagrange interpolation to approximate this function (MA–QRLI). In addition, Mohamad and Sulaiman [6] and Almasieh and Roodaki [8] gave a piecewise collocation solution for FIE by QSIM [6] and TFM [8]. Tohidi [9] used Taylor matrix method to solve 2D LFIE. Lemita et al. [10–11] applied a new process to approach equation (1.1) defined on large interval while Guebbai [12] used regularization and Fourier series to solve it. Ebrahimi and Rashidinia [13], Zhong [14] and Rashidinia et al. [15] investigated the solution of both LFIE and NLFIE using CM [13], integral mean value method (IMVM) [14] and non-polynomial spline method (NPSM) [15]. A novel numerical method depends on the integral mean–value theorem used by Li and Huang [16] to solve equation (1.2). Nadjafi and Heidari [17] solved NLFIE by NKQM. Furthermore, Aziz and Islam [19] determined the solution of it using Haar–wavelets method while

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Maleknejad and Nedaiasl [20] and Maleknejad et al. [21] solved it using sinc–collocation method [20] and triangular functions method [21]. Also, Behiry et al. [22] and Allahviranloo and Ghanbari [23] applied DADM [22] and DHAM [23] to solve equation (1.2). Shiri et al. [25] and Ghanbari et al. [26] determined the solution of FFIE using spectral methods based on Chebyshev polynomials [25] and BPFs [26], respectively. Moreover, Mirzaee [27] investigated the solution of FFIE using BPFs and Taylor series but this paper contained some scientific errors which have been corrected by Sabzevari [28].

Non-polynomial splines have been used to approximate the solutions of several partial differential equations and integral equations such as Fredholm integral equations [15], Volterra integral equations [31–32], hyperbolic equations [33–34], generalized regularized long wave equation [35], Burgers' equations [36–37], parabolic equations [38], Fisher equations [39] and convection–reaction diffusion equation [41] ... etc. In this work, the solution of linear, nonlinear, and fuzzy Fredholm integral equations are obtained using ten non-polynomial cubic splines (TNPCS) method which local truncation error is $O(h^4)$. Also, their results are compared with the existing methods such as MA–QRLI [5], IMVM [14], NPSM [15], NKQM [17] and BPFs [26]. These comparisons conclude that the present method is the best. Also, there are some references [41–53] about the numerical solution of integral equations, the researcher should read them.

This paper is organized as following: In Section 2, the procedure of TNPCS method are presented. In Section 3, the methodology of the present method is applied to LFIE, NLFIE and FFIE. Sections 4 and 5 contain the convergence analysis of TNPCS method and the numerical results of the given examples, respectively. Finally, Section 6 consists of the conclusion.

2. Ten non-polynomial cubic spline method

Firstly, the interval $[a,b]$ is divided into n subintervals with the nodes $t_i = t_0 + ih$, for $i = 0,1,2,\cdots,n$, $t_0 = a$, $t_n = b$ and $h = \frac{t_n - t_0}{n}$. Secondly, let $S_i(t) = y(t)$ be the non-polynomial cubic spline function defined as following:

$$S_i(t) = a_i\left(k_1 e^{\lambda_1\tau(t-t_i)} + k_2 e^{-\lambda_1\tau(t-t_i)}\right) + b_i\left(k_3 e^{\lambda_2\tau(t-t_i)} + k_4 e^{-\lambda_2\tau(t-t_i)}\right) + c_i(t-t_i) + d_i, \quad \text{for } t_i \le t \le t_{i+1} \text{ and } i = 0,1,2,\cdots,n, \tag{2.1}$$

where a_i, b_i, c_i and d_i are the spline coefficients and τ, λ_l and k_m are constants, for $l = 1,2$ and $m = 1,2,3,4$.

Using equation (2.1), we can introduce about ten different formulas for the non-polynomial cubic spline function as shown in Table 1. To define the spline coefficients of equation (2.1), let $y_i = S_i(t_i), y_{i+1} = S_i(t_{i+1}), M_i = S_i''(t_i), M_{i+1} = S_i''(t_{i+1})$ and $\theta = \tau h$, then we get:

$$a_i = \frac{1}{\tau^2}(\alpha_1 M_i + \alpha_2 M_{i+1}),$$

$$b_i = \frac{1}{\tau^2}(\alpha_4 M_i - \alpha_5 M_{i+1}), \tag{2.2}$$

$$c_i = \frac{1}{h}(y_{i+1} - y_i) + \frac{1}{h\tau^2}(\alpha_8 M_{i+1} + \alpha_9 M_i)$$

and $d_i = y_i + \frac{1}{\tau^2\lambda_1^2}((\alpha_{10} - 1)M_i - \alpha_{11}M_{i+1})$, for $i = 0,1,2,\cdots,n-1$, where

$$\alpha_0 = \lambda_1^2\left((k_3+k_4)e^{-\lambda_1\theta}\left(k_1 e^{2\lambda_1\theta} + k_2\right) - (k_1+k_2)e^{-\lambda_2\theta}\left(k_3 e^{2\lambda_2\theta} + k_4\right)\right),$$

$$\alpha_1 = \frac{-1}{\alpha_0}\left(e^{-\lambda_2\theta}\left(e^{2\lambda_2\theta}k_3 + k_4\right)\right),$$

$$\alpha_2 = \left(\frac{k_3+k_4}{\alpha_0}\right),$$

$$\alpha_3 = \lambda_2^2\left(k_1k_3e^{(2\lambda_1+\lambda_2)\theta} + k_2k_3e^{\lambda_2\theta} - k_3(k_1+k_2)e^{(\lambda_1+2\lambda_2)\theta} + k_1k_4\left(e^{(2\lambda_1+\lambda_2)\theta} - e^{\lambda_1\theta}\right) + k_2k_4\left(e^{\lambda_2\theta} - e^{\lambda_1\theta}\right)\right),$$

$$\alpha_4 = \frac{1}{\alpha_3}\left(e^{\lambda_2*\theta}\left(k_1 e^{2\lambda_1*\theta} + k_2\right)\right),$$

$$\alpha_5 = \left(\frac{k_1+k_2}{\alpha_3}\right)e^{(\lambda_1+\lambda_2)\theta},$$

$$\alpha_6 = \lambda_1^2\lambda_2^2\left(k_1k_3e^{(2\lambda_1+\lambda_2)\theta} + k_2k_3e^{\lambda_2\theta} - k_3(k_1+k_2)e^{(\lambda_1+2\lambda_2)\theta} - k_1k_4e^{\lambda_1\theta} + k_1k_4e^{(2\lambda_1+\lambda_2)\theta} - k_2k_4e^{\lambda_1\theta} + k_2k_4e^{\lambda_2\theta}\right),$$

$$\alpha_7 = \frac{1}{\alpha_6}\left(\tau^2\lambda_1^2\lambda_2^2\left(k_2\left(\left(e^{\theta\lambda_2} - e^{\theta(\lambda_1+2\lambda_2)}\right)k_3 - \left(e^{\theta\lambda_1} - e^{\theta\lambda_2}\right)k_4\right) + e^{\theta\lambda_1}k_1\left(-e^{2\theta\lambda_2}k_3 - k_4 + e^{\theta(\lambda_1+\lambda_2)}(k_3+k_4)\right)\right)\right),$$

$$\alpha_8 = \frac{-1}{\alpha_6}\left(-k_3\lambda_1^2(k_1+k_2)e^{(\lambda_1+2\lambda_2)\theta} - k_4\lambda_1^2(k_1+k_2)e^{\lambda_1\theta} + (k_3+k_4)e^{\lambda_2\theta}\left(k_1\lambda_2^2e^{2\lambda_1\theta} + k_2\lambda_2^2 + (k_1+k_2)\left(\lambda_1^2 - \lambda_2^2\right)e^{\lambda_1\theta}\right)\right),$$

$$\alpha_9 = \frac{-1}{\alpha_6}\left(k_2\left(k_4\left(-\left(-1+e^{\theta\lambda_2}\right)\lambda_1^2 + \left(-1+e^{\theta\lambda_1}\right)\lambda_2^2\right) + e^{\theta\lambda_2}k_3\left(\left(-1+e^{\theta\lambda_2}\right)\lambda_1^2 + e^{\theta\lambda_2}\left(-1+e^{\theta\lambda_1}\right)\lambda_2^2\right)\right) + e^{\theta\lambda_1}k_1\left(k_3\left(e^{\theta(\lambda_1+\lambda_2)}\left(-1+e^{\theta\lambda_2}\right)\lambda_1^2 - e^{2\theta\lambda_2}\left(-1+e^{\theta\lambda_1}\right)\lambda_2^2\right) + k_4\left(\lambda_2^2 - e^{\theta\lambda_1}\left(\left(-1+e^{\theta\lambda_2}\right)\lambda_1^2 + \lambda_2^2\right)\right)\right)\right),$$

$$\alpha_{10} = \frac{1}{\alpha_3}(k_3+k_4)\left(\lambda_2^2 - \lambda_1^2\right)\left(k_2e^{\lambda_2\theta} + k_1e^{(2\lambda_1+\lambda_2)\theta}\right)$$

and $\alpha_{11} = \frac{1}{\alpha_3}(k_3+k_4)\left(\lambda_2^2 - \lambda_1^2\right)\left(k_2e^{(\lambda_1+\lambda_2)\theta} + k_1e^{(\lambda_1+\lambda_2)\theta}\right)$.

Table 1
The generating spline function formulas from equation (2.1).

Formula number	Spline function formulas	k_1	k_2	k_3	k_4	λ_1	λ_2
NP.1	$a_i cos(\tau(t-t_i)) + b_i sin(\tau(t-t_i)) + c_i(t-t_i) + d_i$	0.5	0.5	$-0.5i$	$0.5i$	i	i
NP.2	$a_i cos(\tau(t-t_i)) + b_i sinh(\tau(t-t_i)) + c_i(t-t_i) + d_i$	0.5	0.5	0.5	−0.5	i	1
NP.3	$a_i cos(\tau(t-t_i)) + b_i cosh(\tau(t-t_i)) + c_i(t-t_i) + d_i$	0.5	0.5	0.5	0.5	i	1
NP.4	$a_i cos(\tau(t-t_i)) + b_i exp(\tau(t-t_i)) + c_i(t-t_i) + d_i$	0.5	0.5	1	0	i	1
NP.5	$a_i sin(\tau(t-t_i)) + b_i cosh(\tau(t-t_i)) + c_i(t-t_i) + d_i$	$-0.5i$	$0.5i$	0.5	0.5	i	1
NP.6	$a_i sin(\tau(t-t_i)) + b_i exp(\tau(t-t_i)) + c_i(t-t_i) + d_i$	$-0.5i$	$0.5i$	1	0	i	1
NP.7	$a_i sinh(\tau(t-t_i)) + b_i cosh(\tau(t-t_i)) + c_i(t-t_i) + d_i$	0.5	−0.5	0.5	0.5	1	1
NP.8	$a_i sinh(\tau(t-t_i)) + b_i exp(\tau(t-t_i)) + c_i(t-t_i) + d_i$	0.5	−0.5	1	0	1	1
NP.9	$a_i cosh(\tau(t-t_i)) + b_i exp(\tau(t-t_i)) + c_i(t-t_i) + d_i$	0.5	0.5	1	0	1	1
NP.10	$a_i exp(\tau(t-t_i)) + b_i exp(-\tau(t-t_i)) + c_i(t-t_i) + d_i$	1	0	0	1	1	1

Remarks:

i. NP.1, NP.5 and NP.7, defined in Table 1, are the non-polynomial spline functions introduced in [15,31–35,36] and [36–39], respectively.
ii. NP.1 used to solve Fredholm integral equation [15], nonlinear Volterra integral equations [31–32], hyperbolic equations [33], Burgers' equation [34] and 1D quasi–linear hyperbolic equation [35].
iii. NP.5 and NP.7 used to obtain the numerical solution of generalized regularized long wave equation [36].
iv. NP.7 investigated the solution of Burgers–Fisher and coupled nonlinear Burgers' equations [37], 1D quasi–linear parabolic equations [38] and nonlinear Fisher equations [39].

Using the continuity of the first derivative of the defined non-polynomial spline function at the point (t_i, y_i), we have $S'_{i-1}(t_i) = S'_i(t_i)$, then we get the following relation:

$$\beta_0 M_{i-1} + \beta_1 M_i + \beta_2 M_{i+1} = \frac{1}{h^2}(y_{i-1} - 2y_i + y_{i+1}), \quad \text{for} \quad i = 1, 2, 3, \cdots, n-1, \tag{2.3}$$

where

$$\beta_0 = \frac{1}{h^2\alpha_7}\left(-\alpha_9 + \theta\left(-e^{-\theta\lambda_1}\left(e^{2\theta\lambda_1}k_1 - k_2\right)\alpha_1\lambda_1 - e^{\theta\lambda_2}k_3\alpha_4\lambda_2 + e^{-\theta\lambda_2}k_4\alpha_4\lambda_2\right)\right),$$

$$\beta_1 = \frac{1}{h^2\alpha_7}\left(-\alpha_8 + \alpha_9 + \theta(k_1 - k_2)\alpha_1\lambda_1 - e^{\theta\lambda_1}\theta\left(k_1 - e^{-2\theta\lambda_1}k_2\right)\alpha_2\lambda_1 + \theta(k_3 - k_4)\alpha_4\lambda_2 + e^{\theta\lambda_2}\theta k_3\alpha_5\lambda_2 - e^{-\theta\lambda_2}\theta k_4\alpha_5\lambda_2\right)$$

and $\beta_2 = \frac{1}{h^2\alpha_7}(\alpha_8 + \theta(k_1 - k_2)\alpha_2\lambda_1 + \theta(-k_3 + k_4)\alpha_5\lambda_2)$.

The local truncation error (T_i) of TNPCS method can be determined by expanding equation (2.3) by Taylor's approximation about x_i, then we get

$$\begin{aligned} T_i &= (-1 + \beta_0 + \beta_1 + \beta_2)y''_i + h(-\beta_0 + \beta_2)y'''_i \\ &+ \frac{h^2}{12}(-1 + 6\beta_0 + 6\beta_2)y_i^{(4)} + \frac{h^3}{6}(-\beta_0 + \beta_2)y_i^{(5)} \\ &+ \frac{h^4}{360}(-1 + 15\beta_0 + 15\beta_2)y_i^{(6)} + \frac{h^5}{120}(-\beta_0 + \beta_2)y_i^{(7)} \\ &+ \frac{h^6}{20160}(-1 + 28\beta_0 + 28\beta_2)y_i^{(8)} + \cdots, \end{aligned} \tag{2.4}$$

Let $e_1 = -1 + \beta_0 + \beta_1 + \beta_2$, $e_2 = -\beta_0 + \beta_2$ and $e_3 = -1 + 6\beta_0 + 6\beta_2$, then for $e_1 = e_2 = e_3 = 0$, we get
$\beta_0 = \beta_2 = \frac{1}{12}$, $\beta_1 = \frac{5}{6}$,
and $T_i = \frac{1}{240}h^4 y_i^{(6)} + \frac{11}{60480}h^6 y_i^{(8)} + O(h^8)$,
henec, $T_i = O(h^4)$.

Using the natural spline initial conditions, we can take $M_0 = M_n = 0$, then rewriting equation (2.3) in the matrix form, we get:

$$W_1 M = \frac{1}{h^2} W_2 Y$$

and hence

$$M = \frac{12}{h^2} J Y, \tag{2.5}$$

where $J = W_1^{-1} W_2$,

$M = (M_0, M_1, \cdots, M_n)^T$,

$Y = (y_0, y_1, \cdots, y_n)^T$,

$$W_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & & & & 0 \\ 1 & 10 & 1 & 0 & & \cdots & & 0 \\ 0 & 1 & 10 & 1 & 0 & & & 0 \\ & & \vdots & & & \ddots & & \vdots \\ 0 & 0 & & & & 1 & 10 & 1 \\ & & & & & \cdots & & \\ 0 & 0 & & & & & & 1 \end{pmatrix}$$

$$\text{and } W_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & & & 0 \\ 1 & -2 & 1 & 0 & 0 & \cdots & & 0 \\ 0 & 1 & -2 & 1 & 0 & & & 0 \\ & & \vdots & & & \ddots & & \vdots \\ 0 & 0 & & & & 1 & -2 & 1 \\ 0 & 0 & & & & \cdots & & 0 \end{pmatrix}.$$

3. Applying TNPCS on Fredholm integral equations

In this section, the solutions of linear, nonlinear, and fuzzy Fredholm integral Eqs. (1.1) and (1.2) are obtained using TNPCS method introduced in Section 2.

3.1. Linear and nonlinear Fredholm integral equations

Firstly, the solution of LFIE (1.1) is discussed, for this purpose, we substitute from equation (2.2) into (2.1), then we get:

$$\begin{aligned} S_i(t) = y(t) &= \left[\frac{1}{\tau^2}(\alpha_1 M_i + \alpha_2 M_{i+1})\right]\left(k_1 e^{\lambda_1\tau(t-t_i)} + k_2 e^{-\lambda_1\tau(t-t_i)}\right) \\ &+ \left[\frac{1}{\tau^2}(\alpha_4 M_i - \alpha_5 M_{i+1})\right]\left(k_3 e^{\lambda_2\tau(t-t_i)} + k_4 e^{-\lambda_2\tau(t-t_i)}\right) \\ &+ \left[\frac{1}{h}(y_{i+1} - y_i) + \frac{1}{h\tau^2}(\alpha_8 M_{i+1} + \alpha_9 M_i)\right](t - t_i) \\ &+ \left[y_i + \frac{1}{\tau^2\lambda_1^2}((\alpha_{10} - 1)M_i - \alpha_{11}M_{i+1})\right] + O\left(h^4\right), \end{aligned} \tag{3.1}$$

from equation (3.1) into equation (1.1), we have

$$y(t_i) = f(t_i) + \int_a^b \psi(t_i, t)y(t)dt,$$

$$y(t_i) = f(t_i) + \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} \psi(t_i, t)y(t)dt + O\left(h^4\right),$$

$$y(t_i) = f(t_i) + \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} \psi(t_i, t)S_j(t)dt + O\left(h^4\right),$$

$$\begin{aligned} y(t_i) = f(t_i) + \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} \psi(t_i, t)\Bigg[&\left[\frac{1}{\tau^2}(\alpha_1 M_j + \alpha_2 M_{j+1})\right] \\ &\times\left(k_1 e^{\lambda_1\tau(t-t_j)} + k_2 e^{-\lambda_1\tau(t-t_j)}\right) + \left[\frac{1}{\tau^2}(\alpha_4 M_j - \alpha_5 M_{j+1})\right] \\ &\times\left(k_3 e^{\lambda_2\tau(t-t_j)} + k_4 e^{-\lambda_2\tau(t-t_j)}\right) \\ &+ \left[\frac{1}{h}(y_{i+1} - y_i) + \frac{1}{h\tau^2}(\alpha_8 M_{i+1} + \alpha_9 M_i)\right](t - t_j) \\ &+ \left[y_j + \frac{1}{\tau^2\lambda_1^2}((\alpha_{10} - 1)M_j - \alpha_{11}M_{j+1})\right]\Bigg]dt + O\left(h^4\right), \end{aligned}$$

$$y(t_i) = f(t_i) + \sum_{j=0}^{n}\left[\frac{1}{\tau^2}(\alpha_1 M_j)a_{i,j} + \frac{1}{\tau^2}(\alpha_2 M_j)b_{i,j} + \frac{1}{\tau^2}(\alpha_4 M_j)c_{i,j} - \frac{1}{\tau^2}(\alpha_5 M_j)d_{i,j} + \left(\frac{1}{h}y_j + \frac{\alpha_8}{h\tau^2}M_j\right)e_{i,j} + \left(-\frac{1}{h}y_j + \frac{\alpha_9}{h\tau^2}M_j\right)g_{i,j} + \left(y_j + \frac{1}{\tau^2\lambda_1^2}(\alpha_{10}-1)M_j\right)l_{i,j} - \frac{1}{\tau^2\lambda_1^2}(\alpha_{11}M_j)n_{i,j}\right] + O\left(h^4\right), \tag{3.2}$$

where

$$a_{i,j} = b_{i,j+1} = \int_{t_j}^{t_{j+1}} \psi(t_i,t)\left(k_1 e^{\lambda_1\tau(t-t_j)} + k_2 e^{-\lambda_1\tau(t-t_j)}\right)dt,$$

$$c_{i,j} = d_{i,j+1} = \int_{t_j}^{t_{j+1}} \psi(t_i,t)\left(k_3 e^{\lambda_2\tau(t-t_j)} + k_4 e^{-\lambda_2\tau(t-t_j)}\right)dt,$$

$$e_{i,j+1} = g_{i,j} = \int_{t_j}^{t_{j+1}} \psi(t_i,t)(t-t_j)dt$$

and $l_{i,j} = n_{i,j+1} = \int_{t_j}^{t_{j+1}} \psi(t_i,t)dt$.

By assuming $a_{i,n} = b_{i,0} = c_{i,n} = d_{i,0} = e_{i,0} = g_{i,n} = l_{i,n} = n_{i,0} = 0$, $A = a_{i,j}$, $B = b_{i,j}$, $C = c_{i,j}$, $D = d_{i,j}$, $E = e_{i,j}$, $G = g_{i,j}$, $L = l_{i,j}$, $N = n_{i,j}$, $M = (M_0, M_1, \cdots, M_n)^T$, $Y = (y_0, y_1, \cdots, y_n)^T$ and $F = (f_0, f_1, \cdots, f_n)^T$, we get

$$Y = F + \frac{12}{\tau^2}\left[\alpha_1 A + \alpha_2 B + \alpha_4 C - \alpha_5 D + \frac{1}{h}\alpha_8 E + \frac{1}{h}\alpha_9 G + \frac{1}{\lambda_1^2}(\alpha_{10}-1)L - \frac{1}{\lambda_1^2}\alpha_{11}N\right]M + \left[\frac{1}{h}\alpha_7 E - \frac{1}{h}\alpha_7 G + L\right]Y, \tag{3.3}$$

By substituting M from equation (2.5) into (3.4), then we have

$$Y = F + + \frac{12}{\theta^2}\left[\alpha_1 A + \alpha_2 B + \alpha_4 C - \alpha_5 D + \frac{1}{h}\alpha_8 E + \frac{1}{h}\alpha_9 G + \frac{1}{\lambda_1^2}(\alpha_{10}-1)L - \frac{1}{\lambda_1^2}\alpha_{11}N\right]JY + \frac{1}{\tau^2}\left[\frac{1}{h}\alpha_7 E - \frac{1}{h}\alpha_7 G + \tau^2 L\right]Y, \tag{3.4}$$

Let $H_1 = \frac{12}{\theta^2}\left[\alpha_1 A + \alpha_2 B + \alpha_4 C - \alpha_5 D + \frac{1}{h}\alpha_8 E + \frac{1}{h}\alpha_9 G + \frac{1}{\lambda_1^2}(\alpha_{10}-1)L - \frac{1}{\lambda_1^2}\alpha_{11}N\right]$

and $H_2 = \left[\frac{1}{h}\alpha_7 E - \frac{1}{h}\alpha_7 G + L\right]$, then we get

$[I - H_1 J - H_2]Y = F$,

Hence,

$$Y = [I - H_1 J - H_2]^{-1}F. \tag{3.5}$$

Secondly, to obtain the solution of the NLFIE (1.2) using TNPCS method, we need to convert it to a linear equation using the iterative form introduced in [15,32] with a lower solution $y_0(x)$, as following:

$$y_m(x) = f_m(x) + \int_a^b \psi_m(x,t)y_m(t)dt, \quad \text{for} \quad m = 1, 2, \cdots, \tag{3.6}$$

which is the linear Fredholm integral equation of equation (1.2), where

$$f_m(x) = f(t) + \int_a^b \left[\psi(x,t,y_{m-1}(t)) - \psi_y(x,t,y_{m-1}(t))y_{m-1}(t)\right]dt,$$

and $\psi_m(x,t) = \psi_y(x,t,y_{m-1}(t)), \quad \text{for} \quad m = 1, 2, 3, \cdots$ (3.7)

By using equation (3.6) and by repeating the steps from equations (3.1) to (3.4), we have

$$Y_m = [I - H_{1m}J - H_{2m}]^{-1}F_m, \quad \text{for} \quad m = 1, 2, 3, \cdots, \tag{3.8}$$

where

$$Y_m = (y_{0m}, y_{1m}, \cdots, y_{nm})^T,$$

$$F_m = (f_{0m}, f_{1m}, \cdots, f_{n_m})^T,$$

$$H_{1_m} = \frac{12}{\theta^2}\left[\alpha_1 A_m + \alpha_2 B_m + \alpha_4 C_m - \alpha_5 D_m + \frac{1}{h}\alpha_8 E_m + \frac{1}{h}\alpha_9 G_m + \frac{1}{\lambda_1^2}(\alpha_{10}-1)L_m - \frac{1}{\lambda_1^2}\alpha_{11}N_m\right]$$

and $H_{2_m} = \left[\frac{1}{h}\alpha_7 E_m - \frac{1}{h}\alpha_7 G_m + L_m\right]$.

Finally, solving equations (3.5) and (3.8), we get the solutions of equations (1.1) and (1.2) by using TNPCS method.

3.2. Fuzzy Fredholm integral equation

Definition 3.1 ([25,27,29]:). *Any fuzzy number v in the parametric form can be represented by an ordered pair of functions $(\underline{v}(r), \bar{v}(r))$ such that it satisfies the following three conditions*:

i. $\underline{v}(r)$ is a bounded increasing left–continues function,
ii. $\bar{v}(r)$ is a bounded decreasing right–continues function,
iii. $\bar{v}(r) \geq \underline{v}(r)$, $r \in ()$.

Definition 3.2 ([27,29]:). *A fuzzy number is a fuzzy set $v : R^1 \to I = [0,1]$ which satisfies*:

i. v is upper semi–continuous,
iv. $v(x) = 0$, outside the interval $[a,b]$,
v. There are some real numbers $c, d : a \leq c \leq d \leq b$ for which
- $v(x)$ is monotonic increasing on $[a,c]$,
- $v(x)$ is monotonic decreasing on $[d,b]$,
- $v(x) = 1$, $x \in [c,d]$.

Let $v = (\underline{v}, \bar{v})$ and $u = (\underline{u}, \bar{u})$ are two arbitrary fuzzy numbers and $k \in \mathbb{R}$, then for $0 \leq r \leq 1$, their addition and scaler multiplication are defined as following:

i. $(\underline{v+u})(r) = \underline{v}(r) + \underline{u}(r)$,
ii. $(\overline{v+u})(r) = \bar{v}(r) + \bar{u}(r)$,
iii. $\underline{kv}(r) = k\underline{v}(r)$, $\overline{kv}(r) = k\bar{v}(r)$, for $k \geq 0$,
iv. $\underline{kv}(r) = k\bar{v}(r)$, $\overline{kv}(r) = k\underline{v}(r)$, for $k < 0$.

From definition 3.1, we consider the parametric form of $y(x)$ and $f(x)$ are: $y(x,r) = \left(\underline{y}(x,r), \bar{y}(x,r)\right)$ and $f(x,r) = \left(\underline{f}(x,r), \bar{f}(x,r)\right)$,

hence, we can rewrite the parametric form of the fuzzy Fredholm Integral equation (1.1) as introduced in [25] as following:

$$\underline{y}(x,r) = \underline{f}(x,r) + \int_a^b \psi_+(t,x)\underline{y}(t,r)dt - \int_a^b \psi_-(t,x)\bar{y}(t,r)dt, \tag{3.9}$$

and

$$\bar{y}(x,r) = \bar{f}(x,r) + \int_a^b \psi_+(t,x)\bar{y}(t,r)dt - \int_a^b \psi_-(t,x)\underline{y}(t,r)dt, \tag{3.10}$$

where

$$\psi_+(t,x) = \begin{cases} \psi(t,x), & \psi(t,x) \geq 0, \\ 0, & \textit{otherwise}, \end{cases}$$

and $\psi_-(t,x) = \begin{cases} -\psi(t,x), & \psi(t,x) \leq 0, \\ 0, & \textit{otherwise}. \end{cases}$

Rewriting equation (3.1) in the parametric form, we get

$$\begin{aligned}\underline{S_i}(t,r) &= \underline{y}(t,r)\\ &= \left[\frac{1}{\tau^2}\left(\alpha_1\underline{M_i}+\alpha_2\underline{M_{i+1}}\right)\right]\left(k_1e^{\lambda_1\tau(t-t_i)}+k_2e^{-\lambda_1\tau(t-t_i)}\right)\\ &\quad+\left[\frac{1}{\tau^2}\left(\alpha_4\underline{M_i}-\alpha_5\underline{M_{i+1}}\right)\right]\left(k_3e^{\lambda_2\tau(t-t_i)}+k_4e^{-\lambda_2\tau(t-t_i)}\right)\\ &\quad+\left[\frac{1}{h}\left(\underline{y_{i+1}}-\underline{y_i}\right)+\frac{1}{h\tau^2}\left(\alpha_8\underline{M_{i+1}}+\alpha_9\underline{M_i}\right)\right](t-t_i)\\ &\quad+\left[\underline{y_i}+\frac{1}{\tau^2\lambda_1^2}\left((\alpha_{10}-1)\underline{M_i}-\alpha_{11}\underline{M_{i+1}}\right)\right],\end{aligned}\tag{3.11}$$

and

$$\begin{aligned}\underline{S_i}(t,r) &= \bar{y}(t,r)\\ &= \left[\frac{1}{\tau^2}\left(\alpha_1\overline{M_i}+\alpha_2\overline{M_{i+1}}\right)\right]\left(k_1e^{\lambda_1\tau(t-t_i)}+k_2e^{-\lambda_1\tau(t-t_i)}\right)\\ &\quad+\left[\frac{1}{\tau^2}\left(\alpha_4\overline{M_i}-\alpha_5\overline{M_{i+1}}\right)\right]\left(k_3e^{\lambda_2\tau(t-t_i)}+k_4e^{-\lambda_2\tau(t-t_i)}\right)\\ &\quad+\left[\frac{1}{h}\left(\overline{y_{i+1}}-\overline{y_i}\right)+\frac{1}{h\tau^2}\left(\alpha_8\overline{M_{i+1}}+\alpha_9\overline{M_i}\right)\right](t-t_i)\\ &\quad+\left[\overline{y_i}+\frac{1}{\tau^2\lambda_1^2}\left((\alpha_{10}-1)\overline{M_i}-\alpha_{11}\overline{M_{i+1}}\right)\right],\end{aligned}\tag{3.12}$$

where

$\underline{M_i}=\underline{M}(t_i,r), \overline{M_i}=\overline{M}(t_i,r), \underline{y_i}=\underline{y}(t_i,r)$ and $\overline{y_i}=\bar{y}(t_i,r)$.

By substituting $\underline{y}(t,r)$ and $\bar{y}(t,r)$ from equations (3.11) and (3.12) into equations (3.9) and (3.10), then by repeating the steps from equations (3.2) to (3.3), then we have

$$(I-\underline{H_1}J-\underline{H_2})\underline{Y}+(\overline{H_1}J+\overline{H_2})\overline{Y}=\underline{F},\tag{3.13}$$

and

$$(\underline{H_3}J+\underline{H_4})\underline{Y}+(I-\overline{H_3}J-\overline{H_4})\overline{Y}=\overline{F},\tag{3.14}$$

where

$$\underline{H_1}=\frac{12}{\theta^2}\left[\alpha_1\underline{A_1}+\alpha_2\underline{B_1}+\alpha_4\underline{C_1}-\alpha_5\underline{D_1}+\frac{1}{h}\alpha_8\underline{E_1}+\frac{1}{h}\alpha_9\underline{G_1}+\frac{1}{\lambda_1^2}(\alpha_{10}-1)\underline{L_1}-\frac{1}{\lambda_1^2}\alpha_{11}\underline{N_1}\right],$$

$$\underline{H_2}=\left[\frac{1}{h}\alpha_7\underline{E_1}-\frac{1}{h}\alpha_7\underline{G_1}+\underline{L_1}\right],$$

$$\overline{H_1}=\frac{12}{\theta^2}\left[\alpha_1\overline{A_2}+\alpha_2\overline{B_2}+\alpha_4\overline{C_1}-\alpha_5\overline{D_2}+\frac{1}{h}\alpha_8\overline{E_2}+\frac{1}{h}\alpha_9\overline{G_2}+\frac{1}{\lambda_1^2}(\alpha_{10}-1)\overline{L_2}-\frac{1}{\lambda_1^2}\alpha_{11}\overline{N_2}\right],$$

$$\overline{H_2}=\left[\frac{1}{h}\alpha_7\overline{E_2}-\frac{1}{h}\alpha_7\overline{G_2}+\overline{L_2}\right],$$

$$\underline{H_3}=\frac{12}{\theta^2}\left[\alpha_1\underline{A_3}+\alpha_2\underline{B_3}+\alpha_4\underline{C_3}-\alpha_5\underline{D_3}+\frac{1}{h}\alpha_8\underline{E_3}+\frac{1}{h}\alpha_9\underline{G_3}+\frac{1}{\lambda_1^2}(\alpha_{10}-1)\underline{L_3}-\frac{1}{\lambda_1^2}\alpha_{11}\underline{N_3}\right],$$

$$\underline{H_4}=\left[\frac{1}{h}\alpha_7\underline{E_3}-\frac{1}{h}\alpha_7\underline{G_3}+\underline{L_3}\right],$$

$\overline{H_3}=\frac{12}{\theta^2}\left[\alpha_1\overline{A_4}+\alpha_2\overline{B_4}+\alpha_4\overline{C_4}-\alpha_5\overline{D_4}+\frac{1}{h}\alpha_8\overline{E_4}+\frac{1}{h}\alpha_9\overline{G_4}+\frac{1}{\lambda_1^2}(\alpha_{10}-1)\right.$ $\left.\overline{L_4}-\frac{1}{\lambda_1^2}\alpha_{11}\overline{N_4}\right]$ and $\overline{H_4}=\left[\frac{1}{h}\alpha_7\overline{E_4}-\frac{1}{h}\alpha_7\overline{G_4}+\overline{L_4}\right]$.

Finally, solving the system of equations (3.13) and (3.14), then we get the solutions of $\underline{y}(x,r)$ and $\bar{y}(t,r)$ in the form of equations (3.11) and (3.12).

4. Convergence analysis

In this section, the convergence of TNPCS method is discussed. For this purpose, we consider the following lemmas:

Lemma 4.1 ([15,32]**:**). *Let Q be* $(n\times n)$ *matrix with* $\|Q\|_\infty<1$ *then, the matrix* $(I-Q)$ *is invertible and* $\|(I-Q)^{-1}\|_\infty\le\frac{1}{1-\|Q\|_\infty}$.

where I is the identity matrix and $\|Q\|_\infty$ is the infinity norm of the matrix

$Q=(q_{ij})$ which defined as following:

$$\|Q\|_\infty=\max_{1\le i\le n}\left(\sum_{j=1}^{n}|q_{ij}|\right)$$

Lemma 4.2 ([15,32]**:**). *The matrix* $[I-H_1J-H_2]$ *in equation (3.5), is invertible, provided*

$$\|\psi\|_\infty(b-a)\left[\left(\frac{241\left(5+2\sqrt{6}\right)}{20\sqrt{6}\left(48+2\sqrt{6}\right)}\right)\left(\frac{12}{\theta^2}\right)\varphi_3+1\right]<1$$

Proof: For $i=0,1,2,\cdots,n$, we have

$$\|A\|_\infty=\|B\|_\infty\le\|\psi\|_\infty(b-a)\left(\frac{1}{\theta}\right)\left(\left|k_1e^{\lambda_1\theta}-1\right|+\left|1-k_2e^{-\lambda_1\theta}\right|\right),$$

$$\|C\|_\infty=\|D\|_\infty\le\|\psi\|_\infty(b-a)\left(\frac{1}{\theta}\right)\left(\left|k_3e^{\lambda_2\theta}-1\right|+\left|1-k_4e^{-\lambda_2\theta}\right|\right),$$

$$\|E\|_\infty=\|G\|_\infty\le\|\psi\|_\infty(b-a)\left(\frac{h}{2}\right),$$

and $\|L\|_\infty=\|N\|_\infty\le\|\psi\|_\infty(b-a)$,

hence $\|H_1\|_\infty\le\left(\frac{12}{\theta^2}\right)\|\psi\|_\infty(b-a)\varphi_3$,

and $\|H_2\|_\infty\le\|\psi\|_\infty(b-a)$,

$$\varphi_1=\left(\frac{1}{\theta}\right)\left(\left|k_1e^{\lambda_1\theta}-1\right|+\left|1-k_2e^{-\lambda_1\theta}\right|\right)$$

$$\varphi_2=\left(\frac{1}{\theta}\right)\left(\left|k_3e^{\lambda_2\theta}-1\right|+\left|1-k_4e^{-\lambda_2\theta}\right|\right)$$

and $\varphi_3=|\alpha_1+\alpha_2|\varphi_1+|\alpha_4-\alpha_5|\varphi_2+\frac{1}{2}|\alpha_8+\alpha_9|+\frac{1}{\lambda_1^2}|\alpha_{10}-\alpha_{11}-1|$.

The inverse of the diagonal matrix W_1 defined in equation (2.5) can be obtained by using the inversion of general tridiagonal matrices method [30]. For this purpose, let $W_1^{-1}=z_{i,j}$, for $1\le i,j\le n+1$ and as proved in [15,32], we have

i. $z_{i,i}=\frac{m_{i-1}m_{n-i+1}}{m_i}$, for $1\le i\le n+1$,
ii. $z_{1,n+1}=z_{n+1,1}=z_{1,j}=z_{n+1,j}=0$, for $2\le j\le n$,
iii. $z_{i,j}=z_{j,i}$, for $2\le i,j\le n$,
iv. For $2\le i\le n$ and $1\le j\le n+1$,
v. $z_{i,j}=\begin{cases}(-1)^{j-i}\left(\frac{m_{i-1}}{m_{j-1}}\right)z_{j,j}, & i<j,\\ (-1)^{i-j}\left(\frac{m_{n-i+1}}{m_{n-j+1}}\right)z_{j,j}, & i>j,\end{cases}$

where $m_0=1$ and $m_i=\frac{\sqrt{6}}{24}\left[\left(5+\sqrt{24}\right)^i-\left(5-\sqrt{24}\right)^i\right]$.

From equation (2.5), we have $J=W_1^{-1}W_1=w_{i,j}$ and for $1\le i\le n+1$, we obtain

$$w_{i,j} = \begin{cases} z_{i,j+1}, & j=1, \\ z_{i,j+1} - 2z_{i,j}, & j=2, \\ z_{i,j-1} - 2z_{i,j} + z_{i,j+1}, & 3 \le j \le n-1, \\ z_{i,j-1} - 2z_{i,j}, & j=n, \\ z_{i,j-1}, & j=n+1, \end{cases}$$

and hence, $\|J\|_\infty \le \frac{241(5+2\sqrt{6})}{20\sqrt{6}(48+2\sqrt{6})}$.

By using lemma 4.1, the matrix $[I-(H_1J+H_2)]$ is invertible, provided

$$\|H_1J+H_2\|_\infty < 1$$

$$\|\psi\|_\infty(b-a)\left[\left(\frac{241\left(5+2\sqrt{6}\right)}{20\sqrt{6}\left(48+2\sqrt{6}\right)}\right)\left(\frac{12}{\theta^2}\right)\varphi_3+1\right] < 1 \tag{4.1}$$

Theorem 4.1 ([15,32]:). *Let $f(x) \in C^4(I)$ and $\psi(x,t) \in C^4(I\times I)$ such that*

$$\|\psi\|_\infty(b-a)\left[\left(\frac{241\left(5+2\sqrt{6}\right)}{20\sqrt{6}\left(48+2\sqrt{6}\right)}\right)\left(\frac{12}{\theta^2}\right)\varphi_3+1\right] < 1,$$

then consider a unique approximating solution and the obtained error, equation (4.3), satisfies

$$\|\bar{e}\|_{\infty,\Omega} \le \mu\left(h^4\right) \qquad \forall\, \Omega \subset I,$$

where μ is a constant and $I := [a,b]$.

Proof: Suppose that the exact solution of equation (3.5) in the matric form is

$$[I-(H_1J+H_2)]\overline{Y} = F+T, \tag{4.2}$$

where $\overline{Y} = (\bar{y}_0, \bar{y}_1, \bar{y}_2, \cdots, \bar{y}_n)^T$ is the exact solution vector, and $T = (t_0, t_1, t_2, \cdots, t_n)^T$ is the local truncation error vector.

By Subtracting equations (4.2) and (3.5), we get

$$[I-(H_1J+H_2)]\,\bar{e} = T,$$

$$\bar{e} = [I-(H_1J+H_2)]^{-1}T, \tag{4.3}$$

where $\bar{e} = (e_i)^T$ is the column vector of the errors (e_i), $i = 0,1,2,\cdots,n$,

$$e_i = y(t_i) - S_i,$$

$y(t_i)$ is the exact solution and $S_i = y_i$ is the spline solution,

hence, $\bar{e} = (Y - S)$.

From equations (2.4) and (4.3), we have

$$\|T\|_\infty \le \frac{1}{240}h^4\Phi,$$

$$\text{and}\quad \|\bar{e}\|_\infty \le \|(I-(H_1J+H_2))^{-1}\|_\infty \|T\|_\infty, \tag{4.4}$$

where $\Phi = Max_{x_i \le \xi \le x_{i+1}} y^{(6)}(\xi)$ and ξ is a constant.

From equation (4.4), lemmas 4.1 and 4.2, we have

$$\|\bar{e}\|_\infty \le \frac{\|T\|_\infty}{1-\|\psi\|_\infty(b-a)\left[\left(\frac{241(5+2\sqrt{6})}{20\sqrt{6}(48+2\sqrt{6})}\right)\left(\frac{12}{\theta^2}\right)\varphi_3+1\right]},$$

$$\|\bar{e}\|_\infty \le \mu_1\left(h^4\right),$$

$$\text{and hence}\quad \|Y-S\|_\infty \le \mu_1\left(h^4\right), \tag{4.5}$$

where $\mu_1 = \frac{\Phi}{240\mu_2}$, and $\mu_2 = 1 - \|\psi\|_\infty(b-a)\left[\left(\frac{241(5+2\sqrt{6})}{20\sqrt{6}(48+2\sqrt{6})}\right)\left(\frac{12}{\theta^2}\right)\varphi_3+1\right] < 1$.

On the other hand, by using TNPCS method, we can consider $\overline{S}_i(t)$ is the exact solution of equation (3.1) and it's defined as following:

$$\begin{aligned}\overline{S}_i(t) = &\left[\frac{1}{\tau^2}\left(\alpha_1\overline{M}_i + \alpha_2\overline{M}_{i+1}\right)\right]\left(k_1e^{\lambda_1\tau(t-t_i)} + k_2e^{-\lambda_1\tau(t-t_i)}\right)\\ &+\left[\frac{1}{\tau^2}\left(\alpha_4\overline{M}_i - \alpha_5\overline{M}_{i+1}\right)\right]\left(k_3e^{\lambda_2\tau(t-t_i)} + k_4e^{-\lambda_2\tau(t-t_i)}\right)\\ &+\left[\frac{1}{h}\left(\bar{y}_{i+1}-\bar{y}_i\right) + \frac{1}{h\tau^2}\left(\alpha_8\overline{M}_{i+1} + \alpha_9\overline{M}_i\right)\right](t-t_i)\\ &+\left[y_i + \frac{1}{\tau^2\lambda_1^2}\left((\alpha_{10}-1)\overline{M}_i - \alpha_{11}\overline{M}_{i+1}\right)\right] + O\left(h^4\right),\end{aligned} \tag{4.6}$$

where $\bar{y}_i$, $\overline{M}_i$ are the exact solutions of $y(t_i)$ and $y''(t_i)$, respectively.

By Subtracting equations (3.1) and (4.6), we get

$S_i(t) - \overline{S}_i(t) =$

$$\begin{aligned}&\left[\frac{1}{\tau^2}\left(\alpha_1(M_i-\overline{M}_i) + \alpha_2(M_{i+1}-\overline{M}_{i+1})\right)\right]\left(k_1e^{\lambda_1\tau(t-t_i)} + k_2e^{-\lambda_1\tau(t-t_i)}\right)\\ &+\left[\frac{1}{\tau^2}\left(\alpha_4(M_i-\overline{M}_i) - \alpha_5(M_{i+1}-\overline{M}_{i+1})\right)\right]\left(k_3e^{\lambda_2\tau(t-t_i)} + k_4e^{-\lambda_2\tau(t-t_i)}\right)\\ &+\left[\frac{1}{h}\left((y_{i+1}-\bar{y}_{i+1}) - (y_i-\bar{y}_i)\right) + \frac{1}{h\tau^2}\left(\alpha_8(M_{i+1}-\overline{M}_{i+1}) + \alpha_9(M_i-\overline{M}_i)\right)\right](t-t_i)\\ &+\left[y_i + \frac{1}{\tau^2\lambda_1^2}\left((\alpha_{10}-1)(M_i-\overline{M}_i) - \alpha_{11}(M_{i+1}-\overline{M}_{i+1})\right)\right] + O\left(h^4\right),\end{aligned} \tag{4.7}$$

hence, for $i = 0,1,2,\cdots,n-1$ and $t_i \le t \le t_{i+1}$, we get

$$\|S_i(t) - \underline{S_i}(t)\|_\infty \le O\left(h^4\right),$$

$$\text{and hence,}\quad \|S_i - \underline{S_i}\|_\infty \le \mu_3\left(h^4\right), \tag{4.8}$$

where μ_3 is a constant.

Since $\|Y-\overline{S}\|_\infty \le \|Y-S\|_\infty + \|S-\overline{S}\|_\infty$,

therefore, by using equations (4.5) and (4.8), we have

$$\|Y-\overline{S}\|_\infty \le \mu_1\left(h^4\right) + \mu_3\left(h^4\right) \equiv \mu\left(h^4\right), \tag{4.9}$$

where $\mu = \mu_1 + \mu_3$.

Thus, it follows $\|E\| \to 0$ as $h \to 0$, then we explained the convergence of the fourth order TNPCS method.

5. Numerical results

In this section, the numerical solutions of some Fredholm integral equations are determined by using TNPCS method with $\beta_0 = \beta_2 = \frac{1}{12}, \beta_1 = \frac{5}{6}$ and the values of the parameter $\tau = \frac{\theta}{h}$ are determined from β_0, β_1 and β_2 formulas defined in equation (2.3).

Problem 5.1. *Consider the following linear integral equation* [5,15]:

$$y(x) = x + \frac{1}{6}\int_{-1}^{1}(x^4+t^4)y(t)dt, \tag{5.1}$$

and its exact solution is:

$$y(x) = x. \tag{5.2}$$

Table 2 illustrated the absolute errors of equation (5.1) at $n = 20$ which obtained by using TNPCS method with different values of τ and NPSM [15] with $\tau = 1.5$ and 89.888. These results verified that proposed method is better than NPSM and MA–QRLI [5].

Table 2
The absolute errors for problem 5.1 at $h = 0.1$.

$x \rightarrow$	$\tau \downarrow$	0.2	0.4	0.6	0.8	1
NP.1	22.76	3.33×10^{-16}	3.33×10^{-16}	2.22×10^{-16}	0	0
NP.2	−28.38	2.78×10^{-17}	2.78×10^{-16}	0	5.55×10^{-16}	5.55×10^{-16}
NP.3	−28.09	0	1.67×10^{-16}	1.11×10^{-16}	1.11×10^{-16}	0
NP.4	28.66	7.22×10^{-16}	3.33×10^{-16}	5.55×10^{-16}	1.11×10^{-16}	5.55×10^{-16}
NP.5	85.07	4.99×10^{-16}	4.44×10^{-16}	1.11×10^{-16}	0	2.22×10^{-16}
NP.6	52.74	1.11×10^{-16}	0	4.44×10^{-16}	1.11×10^{-16}	2.22×10^{-16}
NP.7	−28.197	0	2.22×10^{-16}	0	5.55×10^{-16}	2.22×10^{-16}
NP.8	28.197	6.94×10^{-16}	2.22×10^{-17}	4.44×10^{-16}	4.44×10^{-16}	0
NP.9	28.197	3.61×10^{-16}	1.67×10^{-16}	9.99×10^{-16}	0	4.44×10^{-16}
NP.10	28.197	2.49×10^{-16}	0	2.22×10^{-16}	3.33×10^{-16}	2.22×10^{-16}
NPSM [15]	1.5	1.1×10^{-15}	2.7×10^{-15}	1.3×10^{-15}	5.5×10^{-16}	2.2×10^{-16}
	89.888	1.1×10^{-16}	2.7×10^{-16}	0	0	0
MA–QRLI [5]	–	5.29×10^{-12}	5.40×10^{-12}	6.10×10^{-12}	8.99×10^{-12}	1.45×10^{-12}

Moreover, the maximum absolute error (MAE) of TNPCS method is $O\left(10^{-16}\right)$ while MAE of MA–QRLI method [5] is $O\left(10^{-12}\right)$. Also, the computations of the different forms of TNPCS are almost the same.

Problem 5.2. *Consider the following linear integral equation* [4]:

$$y(x) = x^2 - 2\int_0^1 (1 + xt)y(t)dt, \tag{5.3}$$

and its exact solution is:

$$y(x) = x^2 - \frac{5}{24}x - \frac{11}{72}. \tag{5.4}$$

A comparison between the exact solution of equation (5.3) and the approximated solutions using IMVM [4] and TNPCS method are shown in Table 3 and Fig. 1 at $n = 50$. This comparison indicates that the TNPCS method is more accurate than IMVM [4]. In addition, MAE of TNPCS method is $O\left(10^{-4}\right)$ while MAE of IMVM [4] is $O\left(10^{-2}\right)$.

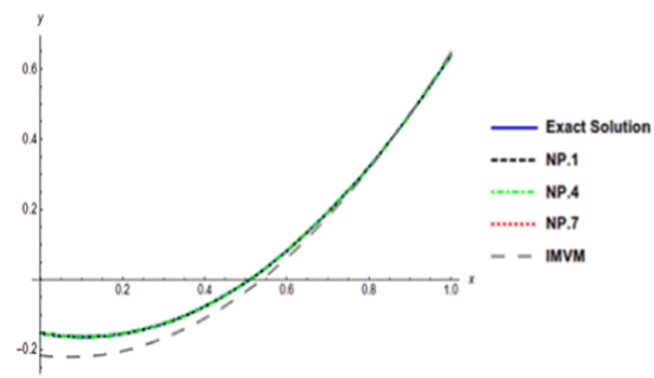

Fig. 1. The solutions of problem 5.2.

Problem 5.3. *Consider the following nonlinear integral equation* [15,17]:

$$y(x) = \text{Sin}(\pi x) + \frac{1}{5}\int_0^1 \text{Cos}(\pi x)\text{Sin}(\pi t)y^3(t)dt, \tag{5.5}$$

with a lower solution is: $y_0(x) = 1 - e^t$, and its exact solution is:

$$y(x) = \text{Sin}(\pi x) + \frac{1}{3}\left(20 - \sqrt{391}\right)\text{Cos}(\pi x), \tag{5.6}$$

The linear integral equation of equation (5.5) in the iterative form is:

$$y_m(x) = \left[\text{Sin}(\pi x) - \frac{2}{5}\int_0^1 \text{Cos}(\pi x)\text{Sin}(\pi t)y_{m-1}^3(t)dt\right] + \frac{3}{5}\int_0^1 \left[\text{Cos}(\pi x)\text{Sin}(\pi t)y_{m-1}^2(t)\right]y_m(t)dt, \quad \text{for } m = 1, 2, 3, \cdots \tag{5.7}$$

Table 4 contains the absolute errors of equation (5.5) that are computed using NP.1 with $h = 0.1, 0.2$ and NKQM [17] with $h = 0.05$. In addition, a comparison between them and the exact solution are shown in Fig. 2 which stated that the TNPCS method is better than NKQM [17]. Also, the solution of the present method can be improved by increasing the values of n and m.

Problem 5.4. *Consider the fuzzy Fredholm integral equation* [26] *with*:

Table 3
The absolute errors for problem 5.2 at $h = 0.02$.

$x \rightarrow$	$\tau \downarrow$	0	0.2	0.4	0.6	0.8	1
NP.1	113.78	4.24×10^{-4}	1.72×10^{-4}	8.00×10^{-5}	3.32×10^{-4}	5.84×10^{-4}	8.35×10^{-4}
NP.3	144.51	4.17×10^{-4}	1.37×10^{-4}	3.38×10^{-5}	5.91×10^{-4}	8.43×10^{-4}	9.86×10^{-4}
NP.4	143.29	4.17×10^{-4}	1.64×10^{-4}	8.84×10^{-5}	3.41×10^{-4}	5.93×10^{-4}	8.46×10^{-4}
NP.6	263.68	4.33×10^{-4}	1.82×10^{-4}	6.94×10^{-5}	3.21×10^{-4}	5.72×10^{-4}	8.23×10^{-4}
NP.7	140.99	4.16×10^{-4}	1.21×10^{-5}	2.41×10^{-4}	4.93×10^{-4}	7.46×10^{-4}	8.36×10^{-4}
NP.10	140.99	4.16×10^{-4}	1.21×10^{-5}	2.41×10^{-4}	4.93×10^{-4}	7.46×10^{-4}	8.36×10^{-4}
IMVM [4]	–	6.23×10^{-2}	4.83×10^{-2}	3.42×10^{-2}	2.01×10^{-2}	6.01×10^{-2}	8.07×10^{-2}

Table 4
The absolute errors for problem 5.3.

x	NKQM [17]	NP.1 ($m = 3$)	
	($n = 20, m = 4$)	($n = 5, \tau = 252.52$)	($n = 10, \tau = 505.04$)
0	4.98×10^{-2}	1.12×10^{-2}	1.39×10^{-4}
0.1	4.74×10^{-2}	2.88×10^{-2}	1.33×10^{-4}
0.2	4.03×10^{-2}	9.06×10^{-3}	1.13×10^{-4}
0.3	2.93×10^{-2}	4.81×10^{-2}	8.20×10^{-5}
0.4	1.54×10^{-2}	3.46×10^{-3}	4.31×10^{-5}
0.5	0	4.89×10^{-2}	4.46×10^{-16}
0.6	1.54×10^{-2}	3.46×10^{-3}	4.31×10^{-5}
0.7	2.93×10^{-2}	3.12×10^{-2}	8.20×10^{-5}
0.8	4.03×10^{-2}	9.06×10^{-3}	1.13×10^{-4}
0.9	4.74×10^{-2}	1.49×10^{-3}	1.33×10^{-4}
1	4.98×10^{-2}	1.12×10^{-2}	1.39×10^{-4}

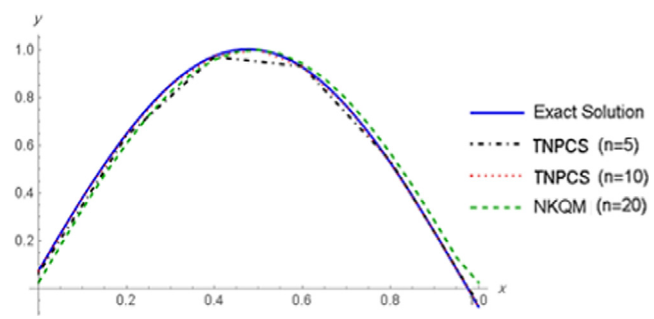

Fig. 2. The solutions of problem 5.3.

$$\psi(x,t) = (2t-1)^2(1-2x), \qquad 0 \le x,t \le 1, \tag{5.8}$$

$$\underline{f}(x,r) = \frac{-1}{3}x^2 + x^2 * r + \frac{1}{3}x + \frac{1}{4}r - \frac{1}{12}, \tag{5.9}$$

$$\text{and}\,\bar{f}(x,r) = \frac{1}{3}x - x^2 * r - \frac{1}{4}r + \frac{5}{3}x^2 + \frac{5}{12}, \tag{5.10}$$

The exact solution of the parametric form $y = (\underline{y}, \bar{y})$ is:

$$\underline{y}(x,r) = rx$$

$$\text{and}\;\; \bar{y}(x,r) = (2-r)x. \tag{5.11}$$

Tables 5 and 6 illustrated the absolute errors of problem 5.4 at $x = 0.5$ using BPFs [26] with $m = 32$ and TNPCS method with $n = 10$ which indicated that the results given by TNPCS method is the best. Furthermore, MAE of TNPCS method is $O\left(10^{-15}\right)$ while MAE of BPFs [26] is $O\left(10^{-2}\right)$. Also, Fig. 3 showed the relation between the numerical solution given by TNPCS method and exact solution at $x = 0.5$.

Problem 5.5. *Consider the LFIE (1.1) [4] with:*

$$\psi(x,t) = \frac{(t-x)\sin(x-t)}{12(1+x)}, \qquad 0 \le x,t \le 1, \tag{5.12}$$

and $f(x)$ is chosen such that its exact solution is:

$$y(x) = \frac{x^5}{20} - \frac{x^4}{12}. \tag{5.13}$$

The absolute errors of problem 5.5 are presented in Table 7 by using TNPCS method and the natural spline interpolation method (NSIM) [4] with $n = 10$ and their MAE is $O\left(10^{-4}\right)$.

Table 5
The absolute errors of $\underline{y}(x,r)$.

r	τ	0	0.2	0.4	0.6	0.8
$\|E\|$ of NP.1	22.76	3.29×10^{-17}	1.55×10^{-17}	3.10×10^{-17}	5.72×10^{-17}	1.44×10^{-16}
$\|E\|$ of NP.3	28.90	1.25×10^{-16}	2.22×10^{-16}	2.78×10^{-16}	3.89×10^{-16}	4.99×10^{-16}
$\|E\|$ of NP.5	85.07	1.25×10^{-16}	9.71×10^{-17}	1.11×10^{-16}	5.55×10^{-17}	5.55×10^{-17}
$\|E\|$ of NP.7	28.197	7.13×10^{-17}	8.33×10^{-17}	8.33×10^{-17}	1.11×10^{-16}	1.11×10^{-16}
$\|E\|$ of NP.9	28.197	3.45×10^{-16}	4.86×10^{-16}	5.83×10^{-16}	6.66×10^{-16}	7.77×10^{-16}
$\|E\|$ of BPFs [26]	–	7.96×10^{-3}	7.74×10^{-3}	1.52×10^{-3}	1.44×10^{-2}	1.12×10^{-2}

Table 6
The absolute errors of $\bar{y}(x,r)$.

r	τ	0	0.2	0.4	0.6	0.8
$\|E\|$ of NP.1	22.76	0	0	5.55×10^{-17}	1.11×10^{-16}	1.14×10^{-16}
$\|E\|$ of NP.3	28.90	2.22×10^{-16}	1.11×10^{-16}	1.08×10^{-18}	1.41×10^{-18}	1.11×10^{-16}
$\|E\|$ of NP.5	85.07	1.33×10^{-15}	1.11×10^{-15}	9.99×10^{-16}	8.89×10^{-16}	6.67×10^{-16}
$\|E\|$ of NP.7	28.197	1.67×10^{-15}	1.44×10^{-15}	1.55×10^{-15}	1.11×10^{-15}	8.88×10^{-16}
$\|E\|$ of NP.9	28.197	6.66×10^{-16}	8.88×10^{-15}	5.55×10^{-16}	5.55×10^{-16}	4.44×10^{-16}
$\|E\|$ of BPFs [26]	–	2.42×10^{-2}	2.74×10^{-2}	3.06×10^{-2}	1.08×10^{-2}	3.09×10^{-2}

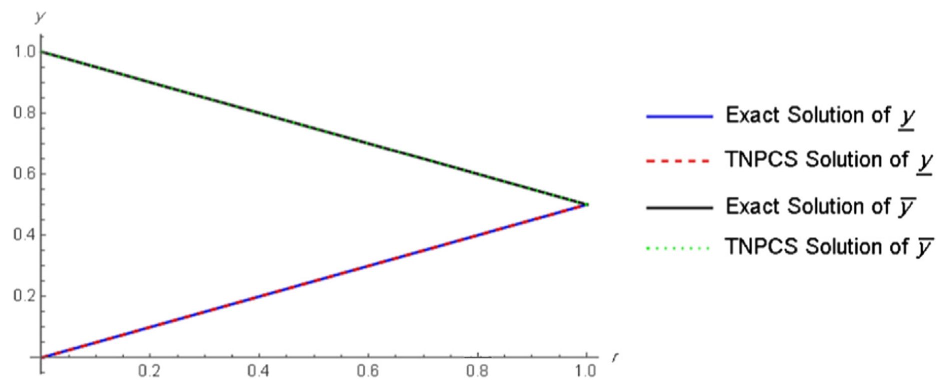

Fig. 3. The solutions of problem 5.4 at: $x = 0.5$ and.$0 \le r \le 1$.

Table 7
The absolute errors for problem 5.5 at $h = 0.1$.

x	0	0.2	0.4	0.6	0.8	1
NP.1	6.36×10^{-6}	5.31×10^{-5}	1.62×10^{-4}	3.20×10^{-4}	5.37×10^{-4}	4.03×10^{-4}
NP.2	6.23×10^{-6}	5.28×10^{-5}	1.60×10^{-4}	3.12×10^{-4}	4.97×10^{-4}	4.30×10^{-4}
NP.3	6.23×10^{-6}	5.28×10^{-5}	1.60×10^{-4}	3.12×10^{-4}	4.95×10^{-4}	4.32×10^{-4}
NP.4	6.23×10^{-6}	5.28×10^{-5}	1.60×10^{-4}	3.12×10^{-4}	4.96×10^{-4}	3.31×10^{-4}
NP.5	3.20×10^{-6}	5.05×10^{-5}	1.58×10^{-4}	3.10×10^{-4}	4.84×10^{-4}	5.37×10^{-4}
NP.6	6.77×10^{-6}	5.33×10^{-5}	1.61×10^{-4}	3.12×10^{-4}	4.85×10^{-4}	4.18×10^{-4}
NP.7	6.19×10^{-6}	5.28×10^{-5}	1.60×10^{-4}	3.12×10^{-4}	4.99×10^{-4}	4.31×10^{-4}
NP.8	6.19×10^{-6}	5.28×10^{-5}	1.60×10^{-4}	3.12×10^{-4}	4.99×10^{-4}	4.31×10^{-4}
NP.9	6.19×10^{-6}	5.28×10^{-5}	1.60×10^{-4}	3.12×10^{-4}	4.99×10^{-4}	4.31×10^{-4}
NP.10	6.19×10^{-6}	5.28×10^{-5}	1.60×10^{-4}	3.12×10^{-4}	4.99×10^{-4}	4.31×10^{-4}
NSIM [4]	2.5×10^{-4}	1.2×10^{-4}	5.5×10^{-5}	1.8×10^{-5}	6.3×10^{-6}	1.2×10^{-5}

6. Conclusion

In this work, a general non-polynomial cubic spline function is presented and used to investigate ten formulas for the cubic spline function by changing the values of some parameters in equation (2.1) as shown in Table 1. In addition, the fourth–order convergence of TNPCS method is derived and then, it's used to obtain the numerical solutions of linear, nonlinear, and fuzzy Fredholm integral equations of second kind. Furthermore, Tables 2–7 and Figs. 1–3 showed that the results given by the proposed method are better than the other mentioned methods.

Notes:

- These computations are obtained using Mathematica 11.1 software.
- The CPU running time used in solving problems 5.1, 5.2 and 5.4, is less than 7 s for each spline function formula (NP.1, NP.2, ..., NP.10) and it's about 25 s for problem 5.5.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Further reading

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